Persistent (co)homology and bottom spectra of Witten and Bismut Laplacians

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In this 6h lecture, I will explain how the low lying spectrum of Witten and Bismut Laplacians are related in some asymptotic regimes with the bar code, in persistent (co)homology, of the potential function $q \mapsto V(q)$. It is based on joint works with D. Le Peutrec, C. Viterbo, F. White and X. Sang. For the Witten Laplacian the semiclassical parameter denoted by $h > 0$ can be interpreted as a temperature while for the Bismut Laplacian two parameters $b > 0$ and $h > 0$ are introduced, where $\frac{1}{b}$ is related to the friction and $h > 0$ is still proportional to the temperature. As $h \to 0^+$ (resp. as $(b, h) \to 0^+$ with some relation between b and h) the logarithms of all the exponentially small eigenvalues of the Witten Laplacians (resp. of the Bismut Laplacian) are given by the bar code of the potential function.
This lecture will be split into several steps:

1. Presentation of persistent homology for a Morse function on a compact manifold. $\mathcal{C}^0$-stability theorem of persistent homology.

2. The semiclassical Witten Laplacian and the (b, h)-dependent version of Bismut Laplacian. Statement of the spectral results.

3. Localization of the analysis in $V_a^b = \{q \in Q, a < V(q) < b\}$ for the semiclassical Witten Laplacian with Dirichlet (at $\{q \in Q, V(q) = a\}$) and Neumann (at $\{q \in Q, V(q) = b\}$) boundary value problems. Proof of the local result when $[a, b]$ contains a single critical value c, $a < c < b$: The exponentially small eigenvalues equal 0.

4. Sketch of the induction proof for N critical values $a < c_1 < \ldots c_N < b$.

5. Main steps of the proof for the Bismut hypoelliptic Laplacian.