Maximal hypoellipticity for scalar Witten Laplacian and related questions

Francis Nier
LAGA,UMR-CNRS 7539
Université Sorbonne Paris Nord
99 av. J.B. Clément
93430 Villetaneuse
email :nier@math.univ-paris13.fr

June 10, 2025

In this talk I will recall how the spectral properties Witten Laplacian acting on functions $\Delta_V^{(0)} = -\Delta + |\nabla V|^2 - \Delta V$, in the euclidean setting with a polynomial potential V, is related with results on maximal hypoellipticity of polynomials of vector fields on a nilpotent Lie algebra. As we explained with B. Helffer in [1], it is actually a microlocal version of maximal hypoellipticity related with the (micro-)hypoellipticity of systems. A nice sufficient condition derived from Helffer-Nourrigat theory (it is actually a criterion for the microlocal maximal hypoellipticity) says that for a polynomial V, the Witten Laplacian $\Delta_V^{(0)}$ has a compact resolvent when none of the limiting polynomials $P(q) = \lim_{n\to\infty} V(\lambda_n q - q_n) - V(q_n)$ of degree less than deg V have a local minimum. This works for the potential $V(q_1, q_2) = -q_1^2 q_2^2$ while it can be checked that 0 belongs to the essential spectrum of $\Delta_V^{(0)}$ when $V(q_1, q_2) = +q_1^2 q_2^2$. The fact that the result depends on the sign $\pm q_1^2 q_2^2$ actually shows the microlocal nature of the problem. Additionally the above criterion provides an algorithm for checking the microlocal maximal hypoellipticity for more general polynomials. Beside this we explored in [1] the question of the compactness of the resolvent for the Kramers-Fokker-Planck operator $K_V = p.\partial_q - \partial_q V.\partial_p + \frac{-\Delta_p + |p|^2}{2}$ and we conjectured that

$$((1 + K_V)^{-1} \text{ compact}) \Leftrightarrow ((1 + \Delta_V^{(0)})^{-1} \text{ compact}) .$$

I will explain what have been the advances made by W.X. Li in [2] and M. Ben Said [3][4] on this question, works which solve the problem for $V(q_1, q_2) = \pm q_1^2 q_2^2$ but not for higher degree polynomials. While doing this I will recall the accurate results obtained with J. Viola and M. Ben Said when deg $V \leq 2$, which contradict in some sense a naive maximal hypoellipticity result.

Although it appears natural from a modelling perspective, the above conjecture has, so far, neither been proven nor disproven by any counterexample. A fresh perspective on this question is therefore awaited.

References

[1] B. Helffer, F. Nier *Hypoelliptic Estimates and Spectral Theory for Fokker-Planck Operators and Witten Laplacians.* Lect. Notes in Math. 1862 (Springer-Verlag, Berlin, 2005).

[2] W.X. Li Compactness criteria for the resolvent of the Fokker-Planck operator. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 18, No. 1, 119-143 (2018).

[3] M. Ben Said Global subelliptic estimates for Kramers-Fokker-Planck operators with some class of polynomials J. Inst. Math. Jussieu 21, No. 2, 675-711 (2022).

[4] M. Ben Said Kramers-Fokker-Planck operators with homogeneous potentials. Math. Methods Appl. Sci. 45, No. 2, 914-927 (2022).

[5] M. Ben Said, F. Nier, J. Viola Quaternionic structure and analysis of some Kramers-Fokker-Planck operators. Asymptotic Anal. 119, No. 1-2, 87-116 (2020).